A crystallographic representation of the braid group

1989 J. Phys. A: Math. Gen. 22 L871

(http://iopscience.iop.org/0305-4470/22/18/003)

J. Phys. A: Math. Gen. 22 (1989) L871. Printed in the UK

LETTER TO THE EDITOR

A crystallographic representation of the braid group

Henri Bacry

Centre de Physique Théorique 2, CNRS, Luminy-Case 907, F-13288, Marseille Cedex 9, France and Faculté des Sciences de Luminy, Marseille, France

Received 26 June 1989

Abstract. We give a homomorphism between the braid group B_n and the symmetry group of a face-centred cubic crystal in an $(n+1)$-dimensional Euclidean space. This representation suggests a continuous family of other realisations of B_n.

Denote by β_i $(i=1,2,\ldots,n)$ the generators of the braid group B_n defined by the relations

$$\beta_i\beta_{i+1}\beta_i=\beta_{i+1}\beta_i\beta_{i+1} \qquad \beta_i^2=1$$

$$\beta_i\beta_j=\beta_j\beta_i \qquad \text{for } |i-j|\neq 1.$$

Let $e_1, e_2, \ldots, e_{n+1}$ be an orthonormal basis of a real $(n+1)$-dimensional Euclidean space. We have the following action of the braid group on a vector

$$x=x_1e_1+\ldots+x_{n+1}e_{n+1}$$

$$\rho(\beta_i)(x)=x_1e_1+\ldots+x_{i+1}e_i+(x_i+1)e_{i+1}+\ldots+x_{n+1}e_{n+1}.$$

This action defines a homomorphism of the braid group to a crystallographic group Γ_{n+1}. This group is the symmetry group of a face-centred cubic crystal in an $(n+1)$-dimensional Euclidean space.

We have the following property:

$$\rho((\beta_i\beta_{i+1})^3)=\rho((\beta_{i+1}\beta_i)^3)=T_i \qquad i=1,2,\ldots,n-1$$

with T_i being a translation acting as

$$T_i(x)=x+2(e_i+e_{i+1}+e_{i+2}).$$

One readily sees that the point group of the crystallographic group is generated by the transpositions of the basis vectors. It follows that the point group is isomorphic to the permutation group S_{n+1}.

This crystallographic realisation suggests an infinite set of other realisations labelled by two arbitrary parameters t and u. They are given by the formulae

$$\sigma_{t,u}(\beta_i)(x_1e_1+\ldots+x_{n+1}e_{n+1})$$
$$=x_1e_1+\ldots+tx_{i+1}e_i+(tx_i+t^{-i+1}u)e_{i+1}+\ldots+x_{n+1}e_{n+1}.$$

Here too, the elements $(\beta_i\beta_{i+1})^3$ and $(\beta_{i+1}\beta_i)^3$ act in the same way. It follows that it is not a faithful representation of the braid group.

If we set $t=u=1$, we are back to the above crystallographic representation ρ. Another case of interest is obtained when we choose $t=\pm\mathrm{i}$; in that case, the three elements $(\beta_i\beta_{i+1})^3$, $(\beta_{i+1}\beta_i)^3$ and β_i^4 act as the identity operator, whatever the value of the other parameter u. More generally, if t is a primitive root of 1 $(t^{4n}=1)$, one has $(\beta_i\beta_{i+1})^{3n}$, $(\beta_{i+1}\beta_i)^{3n}$ and β_i^{4n} acting as the identity, whatever the value of u.

0305-4470/89/180871+01\$02.50